J. Fluid Mech. (1965), *vol.* 21, *part* 1, *pp.* 33–45
Printed in Great Britain

On backward boundary layers and flow in converging passages

By SYDNEY GOLDSTEIN
The Division of Engineering and Applied Physics,
Pierce Hall, Harvard University, Cambridge, Massachusetts, U.S.A.

(Received 20 April 1964)

In a backward boundary layer the fluid has, in the mathematical model, been flowing along a solid wall through an infinite distance. The co-ordinate distance x along the boundary is measured upstream, and the velocity U of the flow outside the boundary layer is taken as negative. The main application is to flow in converging passages.

The existence of similar solutions is considered, with emphasis on the correct asymptotic behaviour for large values of the stretched co-ordinate normal to the wall. This emphasis is shown to be necessary in considering backward boundary layers.

For two-dimensional flow in converging passages the requirement that a boundary layer should be possible for vanishingly small viscosity with a potential core flow is shown to lead directly to Hamel's spirals as the shape of the boundary streamlines.

Flow in axisymmetric converging passages is considered. For flow in a cone there is no limit as the viscosity tends to zero, and no potential core flow with a boundary layer is possible. The nature of a solution of the Navier–Stokes equations for laminar flow is considered.

1. The boundary-layer equation, and the condition 'at infinity'

For a fluid of constant density ρ and constant kinematic viscosity ν, the equation for the first approximation to two-dimensional laminar flow in a boundary layer is

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = U\frac{dU}{dx}+\nu\frac{\partial^2 u}{\partial y^2}. \tag{1}$$

Here x is distance measured along the bounding solid surface from some origin, y is distance normal to the surface, and u and v are the components of the fluid velocity in the directions of x and y increasing. U is usually described as the velocity in the main stream just outside the boundary layer.

From the equation of continuity there is a stream function $\psi(x,y)$ such that

$$u = \frac{\partial\psi}{\partial y}, \quad v = -\frac{\partial\psi}{\partial x}. \tag{2}$$

The boundary conditions at impermeable walls for no slip, which are the usual conditions, may be taken as

$$\psi = \frac{\partial\psi}{\partial y} = 0 \quad \text{at} \quad y = 0. \tag{3}$$

The boundary condition 'at infinity' is sometimes stated to be $u/U \to 1$ as $y \to \infty$. However, it is here necessary to be precise about this boundary condition. This condition is here stated as

$$Y^N(u/U - 1) \to 0 \quad \text{as} \quad Y \to \infty \tag{4}$$

for any real N, where

$$Y = y/\nu^{\frac{1}{2}}. \tag{5}$$

The co-ordinate y does not tend to infinity in the boundary layer. A stretched co-ordinate Y must be used in the boundary layer, and the limit taken is for variations in ν for non-zero y, with ν tending to zero, and not for variations in y. For non-zero y, $Y \to \infty$ as $\nu \to 0$. As $\nu \to 0$ the boundary layer becomes a vortex sheet at the boundary, and any non-zero y is eventually in the inviscid flow outside the boundary layer. Certainly the value of U is taken for small y, in fact for vanishingly small y, but y is taken vanishingly small after $\nu \to 0$, and the matching may be roughly said to be made on the other side of the vortex sheet from that on which the solid boundary lies. In elementary boundary-layer theory, which is all that will be used in this paper, this makes no difference to the actual calculations, but the point is of importance in the further developments which will be mentioned later in connexion with the second remark about the boundary condition at infinity.

Also $(u/U - 1)$ must tend to zero faster than any negative power of Y; the error will, in fact, be exponentially small for large Y. The physical reason is that, if this were not so, there would be extra vorticity in the main flow, over and above any produced before the stream arrives at the surface considered, which would not be exponentially small. But this extra vorticity is produced only at the wall; it then diffuses and is convected with the stream, and in a continuum the result of these processes must be to make this extra vorticity exponentially small in Y when Y is large. This emendation does enter even in elementary boundary-layer theory, and it has been known for a long time that in some cases such a condition is needed to make the solution unique. Hartree discovered this, for example, in calculating the solutions for $U = cx^m$, when m is negative and $0 > m > -0{\cdot}0904$. (See, for example, the reference in *Modern Developments in Fluid Dynamics*, vol. 1, p. 141.)

To explain why the condition is necessary mathematically, consider how boundary-layer theory would be extended to find asymptotic approximations for small ν to a solution of the Navier–Stokes equations to a higher order (Kaplun 1954; Lagerstrom & Cole 1955). To explain the procedure, it will here suffice to consider the two-dimensional flow of a fluid of constant properties along a plane surface. Outside the boundary layer, the 'natural' co-ordinates x and y would be used, and derivatives with respect to these variables would be considered bounded when $\nu \to 0$, in the usual way. Inside the boundary layer, derivatives with respect to y would become infinite as $\nu \to 0$, and the stretched variable Y would be used, derivatives with respect to Y being considered as bounded. Since $u = \nu^{-\frac{1}{2}}\,\partial\psi/\partial Y$, the stream function must also be of order $\nu^{\frac{1}{2}}$ in the boundary layer. Expansions in powers of $\nu^{\frac{1}{2}}$ might first be tried, with

$$\psi_{\text{outside}} = g_0(x,y) + \nu^{\frac{1}{2}} g_1(x,y) + \nu g_2(x,y) + \ldots, \tag{6}$$

$$\psi_{\text{inside}} = \nu^{\frac{1}{2}}[f_0(x, Y) + \nu^{\frac{1}{2}} f_1(x, Y) + \nu f_2(x, Y) + \ldots]. \tag{7}$$

It is known that expansions of these forms are not sufficiently general, but the simplest possible case will serve to illustrate the matter under discussion.

It is postulated that expansions such as (6) and (7) are asymptotic expansions in the sense of Poincaré for ν tending to zero.

The expansions (6) and (7) must be completely matched for small values of y but large values of Y. It is unnecessary here to enter into details. It is sufficient to remark that the asymptotic expansions of $f_0, f_1, f_2, \ldots$ for large Y provide the values of $g_1(x, 0)$, $g_2(x, 0)$, $g_3(x, 0)$, ..., respectively, and so provide boundary conditions for the determination of the $g(x, y)$. This last statement was made on the assumption that no negative powers of Y occur in the asymptotic expansions of $f_0, f_1, \ldots$. If there is a term in Y^{-N} for positive N, this would be a term in $\nu^{\frac{1}{2}N}y^{-N}$, and it would have to be taken into account in the matching process when terms of order $\nu^{\frac{1}{2}N}$ in ψ_{outside} are considered, and $g_N(x, y)$ would have to behave like y^{-N} for small y. Now if no disturbances are introduced upstream, it is not difficult to see that $g_1, g_2, \ldots$ are all harmonic if g_0 is harmonic, or if the vorticity, $-\nabla^2 g_0$, in the original inviscid stream is constant. There is certainly no harmonic function which becomes infinite along any finite portion of the x-axis. (A simple and rigorous proof was once pointed out to me by Prof. Shiffer.) I do not know if proofs of non-existence have been published for more general cases. In particular, I do not know if a proof of non-existence has been published which would cover the case of axisymmetric flow, to which a boundary condition such as (4) will also be applied. It seemed worthwhile to point out one way in which a looser condition could lead to a mathematical impossibility. Henceforward, a boundary condition such as (4) will be postulated. For the two-dimensional case, only equations (1)–(5) will be needed.

2. Similar solutions

Consider again the known results for similar solutions for two-dimensional flow. Suppose that, if at each section x of the boundary layer we scale y correctly, the curves of u/U become the same, i.e. inside the boundary layer

$$\frac{u(x,y)}{U(x)} = \text{a function of } \eta, \text{ where } \eta = \frac{y}{\nu^{\frac{1}{2}}}g(x), \tag{8}$$

for some g. This requires that the stream function inside the boundary layer should be given by

$$\psi = \nu^{\frac{1}{2}}\frac{U(x)}{g(x)}F(\eta), \tag{9}$$

and then

$$\frac{u}{U} = F'(\eta). \tag{10}$$

It is easy to see that this kind of solution is possible if

$$U = cx^m \quad \text{or} \quad ce^{kx} \quad \text{or} \quad ce^{-kx}, \tag{11}$$

where c, m, and k are constants, and x is distance along the boundary from any arbitrary point of it (Goldstein 1939). If c is positive and x is measured downstream, we have the usual forward boundary layers. With $U = cx^m$ $(c > 0)$ and

$$g(x) = (U/x)^{\frac{1}{2}} = c^{\frac{1}{2}}x^{\frac{1}{2}(m-1)}, \tag{12}$$

the equation for F is the well-known equation of Falkner & Skan (1930),

$$F''' + \tfrac{1}{2}(m+1)FF'' - mF'^2 + m = 0, \tag{13}$$

solutions of which, for $m+1 > 0$, were tabulated by Hartree (1937) after the transformation

$$Y = [\tfrac{1}{2}(m+1)]^{\frac{1}{2}}\eta, \quad f(Y) = [\tfrac{1}{2}(m+1)]^{\frac{1}{2}}F(\eta). \tag{14}$$

(Note that the definition of Y here, and from now on, is different from that in equation (5).) The equation for f is

$$f''' + ff'' - \beta(f'^2 - 1) = 0, \tag{15}$$

with

$$\beta = \frac{2m}{m+1}. \tag{16}$$

If $m+1 < 0$, we put

$$Y = [-\tfrac{1}{2}(m+1)]^{\frac{1}{2}}\eta, \quad f(Y) = [-\tfrac{1}{2}(m+1)]^{\frac{1}{2}}F(\eta), \tag{17}$$

and then

$$f''' - ff'' + \beta(f'^2 - 1) = 0, \tag{18}$$

with the same β. Primes on f denote derivatives with respect to Y, and on F with respect to η.

Of course (15) and (18) are reducible to the same equation if imaginary variables are allowed. However, we are here particularly concerned with the existence of real solutions in terms of a real independent variable, so it seems advisable not to use imaginary variables. Also (18) becomes (15) if the sign of f is changed, but then the last boundary condition in (25) below is altered. For our present purposes, it is immaterial whether f or $-f$ be used as dependent variable.

Similar remarks apply to other pairs of equations in the paper, for example (13) and (28), (19) and (29).

If $m+1 = 0$, the equation for $F(\eta)$ is

$$F''' + F'^2 - 1 = 0. \tag{19}$$

With $U = ce^{kx}$ $(c > 0, k > 0)$ and

$$g(x) = (\tfrac{1}{2}kc)^{\frac{1}{2}}\, e^{\frac{1}{2}kx}, \tag{20}$$

$$F''' + FF'' - 2(F'^2 - 1) = 0, \tag{21}$$

and with $U = ce^{-kx}$ $(c > 0, k > 0)$ and

$$g(x) = (\tfrac{1}{2}kc)^{\frac{1}{2}}\, e^{-\frac{1}{2}kx}, \tag{22}$$

$$F''' - FF'' + 2(F'^2 - 1) = 0. \tag{23}$$

The boundary conditions are

$$F(0) = 0, \quad F'(0) = 0, \quad \eta^N(F' - 1) \to 0 \quad \text{as} \quad \eta \to \infty, \tag{24}$$

or

$$f(0) = 0, \quad f'(0) = 0, \quad Y^N(f' - 1) \to 0 \quad \text{as} \quad Y \to \infty, \tag{25}$$

for any real N.

Equation (21) is just equation (15) with $\beta = 2$ ($m \to \infty$), and equation (23) is just equation (18) with $\beta = 2$ ($m \to -\infty$).

3. Backward boundary layers

It was stated in § 2 that similar solutions are possible if $U = cx^m$ or $c\,e^{kx}$ or $c\,e^{-kx}$, and c was taken as positive in § 2. There are also similar solutions if U is given by one of these formulae with c negative, or if

$$U = c(l \pm x)^m,$$

where c, l and m are constants, so long as this expression is real. Then g may be taken to be given by

$$g = |c|^{\frac{1}{2}}(l \pm x)^{\frac{1}{2}(m-1)}$$

if this is real and the equation for F that takes the place of (13) is

$$F''' \pm \operatorname{sgn} c\{\tfrac{1}{2}(m+1)\,FF'' - mF'^2 + m\} = 0.$$

Attention will here be restricted to the cases when

$$U = -c\,x^m \quad \text{or} \quad -c\,e^{kx} \quad \text{or} \quad -c\,e^{-kx}, \tag{26}$$

with $c > 0$, $k > 0$. Thus U is here negative and x is measured upstream. The main application is to flow in converging passages, where, in the mathematical models, the fluid enters at infinity with infinitely slow velocity, and moves in a converging passage towards the intersection of the walls, which is taken as origin. In such cases, in the mathematical models, viscosity has had a sensible effect on the fluid in the boundary layer through an infinite distance. The name 'backward boundary layers' will here be used for such cases.

With $U = -cx^m$, since all quantities must be kept real $g(x)$ is taken as

$$g(x) = (|U|/x)^{\frac{1}{2}} = c^{\frac{1}{2}}x^{\frac{1}{2}(m-1)}, \tag{27}$$

and the equation for $F(\eta)$ is

$$F''' - \tfrac{1}{2}(m+1)\,FF'' + mF'^2 - m = 0. \tag{28}$$

When $m+1 > 0$, Y and $f(Y)$ are defined as before by (14), and the result is that f satisfies equation (18), which was the equation it satisfied for $U > 0$, $m+1 < 0$. Thus f satisfies (18) if $(m+1)\,U < 0$.

Similarly, if $m+1 < 0$, Y and $f(Y)$ are defined as before by (17), and then f satisfies equation (15), which was the equation it satisfied for $U > 0$, $m+1 > 0$. Thus f satisfies (15) if $(m+1)\,U > 0$.

If $m+1 = 0$, the equation for $F(\eta)$ is

$$F''' - F'^2 + 1 = 0. \tag{29}$$

With $U = -c\,e^{kx}$ ($c > 0, k > 0$) and $g(x)$ given by (20), F satisfies equation (23), whereas with $U = -c\,e^{-kx}$ and $g(x)$ given by (22), F satisfies equation (21). Equations (21) and (23) are, as previously noted, the limiting cases of (15) and (18) when $\beta = 2$.

For backward boundary layers the boundary conditions (24) and (25) still apply. The differential equations with which we are concerned for similar solutions are then (15), (18), (19) and (29), with the boundary conditions (24) or (25). The question at issue is whether solutions exist.

β is given by (16). Note that $0 < \beta < 2$ for $m > 0$, that $\beta < 0$ for $-1 < m < 0$, and that $\beta > 2$ for $m+1 < 0$.

4. Existence and uniqueness

We are very much concerned with the nature of the asymptotic behaviour of f as $Y \to \infty$, or of F as $\eta \to \infty$. The last of each of the sets of boundary conditions (24) and (25) must be stressed.

In a rough, non-rigorous manner, possible non-existence of solutions may be discussed as follows. If there is a solution, since $f'(Y) = 1 + o(Y^{-N})$ as $Y \to \infty$, $f(Y) \sim Y + b$, where b is a constant. Put

$$Y + b = \zeta, \quad f = \zeta + g(\zeta). \tag{30}$$

Then $g(\zeta) = o(\zeta^{-N})$ as $\zeta \to \infty$; in fact $g(\zeta)$ will be exponentially small as $\zeta \to \infty$.

Consider equation (18). It becomes

$$g''' - \zeta g'' + 2\beta g' = gg'' - \beta g'^2, \tag{31}$$

where primes now denote derivatives with respect to ζ. For large ζ the right-hand side will be exponentially small compared with the left-hand side, and the equation becomes approximately

$$g''' - \zeta g'' + 2\beta g' = 0. \tag{32}$$

This is a second-order linear equation for g'. Two independent solutions are $\exp(\frac{1}{4}\zeta^2) D_{2\beta}(\zeta)$ and $i^{2\beta+1} \exp(\frac{1}{4}\zeta^2) D_{-2\beta-1}(i\zeta)$, where $D_n(\zeta)$ is Weber's parabolic cylinder function (Whittaker & Watson 1920, sect. 16.5 *et seq.*). Thus the asymptotic expansions for large ζ of two independent solutions of (32) are

$$\zeta^{2\beta}\left[1 - \frac{2\beta(\beta-1)}{2\zeta^2} + \dots\right] \tag{33}$$

and

$$e^{\frac{1}{2}\zeta^2}\zeta^{-2\beta-1}\left[1 + \frac{(-2\beta-1)(-2\beta-2)}{2\zeta^2} + \dots\right], \tag{34}$$

respectively. Neither of these satisfies the condition $g' = o(\zeta^{-N})$ for every N; we need a negative exponential. We conclude that there are no admissible solutions of (18).

With a less stringent boundary condition at infinity, if algebraic error terms—negative powers of Y—had been allowed, the expansion in (33) would have been allowable for negative β. For forward boundary layers, (18) applies with $m+1 < 0$, and therefore with $\beta > 2$, so this is irrelevant. But for backward boundary layers, (18) applies with $m+1 > 0$, so with the less stringent boundary condition it might have been thought that an admissible solution could exist for $-1 < m < 0$, for which $\beta < 0$. With the necessary and more stringent boundary condition, we now see that there is no solution for a backward boundary layer with $m+1 > 0$.

If we consider equation (15) in the same way, we find the following two asymptotic expansions for large ζ of two independent solutions for g':

$$e^{-\frac{1}{2}\zeta^2}\zeta^{-2\beta-1}\left[1 - \frac{(-2\beta-1)(-2\beta-2)}{2\zeta^2} + \dots\right] \tag{35}$$

and

$$\zeta^{2\beta}\left[1 + \frac{2\beta(2\beta-1)}{2\zeta^2} + \dots\right]. \tag{36}$$

The former of these may give an admissible solution for any β. But whereas it appears safe to conclude that there is no admissible solution if this rough method so indicates, it is not safe to conclude that an admissible solution exists if this rough method shows that it may. For one thing, a solution may exist which is not real, and therefore not admissible. Again considering the matter in a loose manner, when we compute a solution we may integrate forwards from the origin ($Y = 0$) with one undetermined constant, and backwards from infinity with two undetermined constants, and make f, f' and f'' continuous at some value of Y. There would be three equations for three unknown constants, but the solution may not be real. In fact, Hartree discovered, numerically, that (15) has a real solution for $\beta \geqslant \beta_0 = -0{\cdot}1988$; for $\beta < \beta_0$, there are no real solutions; and for values of β just greater than β_0, $f''(0)$ is of order $(\beta-\beta_0)^{\frac{1}{2}}$. Also, for $m = -1/3$, $\beta = -1$, a method used by Mills (1938) may be used to find explicitly the solution with the given boundary conditions in terms of Weber's parabolic cylinder functions, and the solution is not real.

Equations (19) and (29) may be solved explicitly. The solution of (29) with the given boundary conditions is the known solution for the boundary layer in converging flow between non-parallel plane walls, with $U = -c/x$. It is found that there is no solution of (19) satisfying the given boundary conditions.

Equations (19) and (29) may also be discussed by the approximate method used above for equations (15) and (18). When (19) is discussed in this way, the approximate equation for g' is

$$g''' + 2g' = 0, \tag{37}$$

so there is no non-zero solution for g' which $\to 0$ as $\zeta \to \infty$.

When (29) is discussed in this way, the approximate equation for g' is

$$g''' - 2g' = 0, \tag{38}$$

and for the admissible solution $g' \sim$ const. $e^{-2\sqrt{\zeta}}$. The exact solution of (29) does indeed exhibit this behaviour.

The above rough procedure was used by Goldstein (1939) to 'show' the non-existence of solutions of (18) with $\beta = 2$—i.e. of (23). The non-existence was rigorously and neatly proved by Hardy (1939), by a method which may be extended to the case of any positive β.

Weyl (1942) proved the existence of solutions of (15) for $\beta > 0$. Coppel (1960) reconsidered this question, and proved that, for all non-negative $f(0)$ and $f'(0)$, there is a solution for which f'' is one-signed, and if f'' must be one-signed the solution is unique and the asymptotic behaviour is correctly given by the rough procedure above.

Through the courtesy of Prof. James Serrin, of the Department of Mathematics of the University of Minnesota, I have recently seen a copy of his lecture notes on *Mathematical Aspects of Boundary Layer Theory*. These contain a derivation of equations (15) and (18) for similar solutions for negative U, and an interesting and full discussion of work on the existence and uniqueness of solutions of equations (15) and (18) under the conditions $f(0) = f'(0) = 0$, $f'(\infty) = 1$, $0 \leqslant f'(Y) \leqslant 1$. A discussion of the asymptotic behaviour of f' as $Y \to \infty$ in the various cases was not included. Additional references are also given in these lecture notes, especially the references to the work of Iglisch (1953, 1954) on the existence and uniqueness of solutions of (15) with $\beta > 0$.

The known results when $U \propto x^m$ may now be summarized as follows.

For forward boundary layers a real solution exists when $m > -0{\cdot}0904$ (equation (15) with $-0{\cdot}1988 < \beta < 2$); and with the stringent boundary condition at infinity and $f''(0)$ positive (i.e. with positive skin-friction) the solution is unique (and there is no back flow). For $m = -0{\cdot}0904$ ($\beta = -0{\cdot}1988$); $f''(0) = 0$. For $m \leqslant -1$ (equations (18) and (19)), no solutions exist. For $-1 < m < -0{\cdot}0904$ (equation (15) with $\beta < -0{\cdot}1988$), no real solutions exist; any solutions which exist must be complex.

For backward boundary layers there are no solutions for $m > -1$ (equation (18)). For $m \leqslant -1$ (equation (15) with $\beta > 2$ and equation (29)) a real solution exists, which is unique with $f''(0)$ positive (positive skin-friction and no back flow).

Although no use is made here of solutions for which $f''(0)$ is negative, it may be mentioned that real solutions have been found to exist for forward boundary layers for $-0{\cdot}0904 < m < 0$ (Stewartson 1954) and for backward boundary layers with $m = -1$ (Ackerberg 1962).

Similar solutions are known to be useful because they provide standards of comparison for approximate methods. In what follows their use is different; if the whole solution is not a similar solution, the results from considering similar solutions provide a guide on the way an expansion should start. The results will be used in this way, for example, to consider converging flow in a cone.

5. Applications to two-dimensional flow in converging passages

For flow in a converging passage, since m must be $\leqslant -1$, there must be a singularity where the walls would meet, at the origin. The least singularity possible for a flow which would allow a boundary layer to exist and bring the slip at the wall to zero is when $m = -1$. If a solution without a singularity is impossible, and no external agency is interfering at the position of the singularity, it is now postulated that the solution with the smallest allowable singularity will be the best description of the physical facts. Thus, if we now write s for distance along a streamline from the intersection of the two bounding streamlines, and q for the resultant velocity, and look for an irrotational flow outside the boundary layers, the flows in contracting passages for which there will, for vanishingly small ν, be such potential flows are those for which

$$q \propto 1/s \tag{39}$$

along the bounding streamlines (on the assumption that a higher singularity is not forced in the theory by considerations of continuity, as it may be if the bounding streamlines intersect at a cusp). The shape of the streamlines may be determined from this condition; this was done by Hadley Smith (unpublished), and they are, as expected, Hamel's equiangular spirals. A long time ago Hamel (1916) sought the viscous flows for which the streamlines coincide with the streamlines of an inviscid potential flow without the actual viscous motion itself being irrotational; he found the streamlines to be equiangular spirals. The flow between non-parallel straight walls is a particular case.

Thus we may use boundary-layer theory to ask the right question about inviscid potential flows, and find Hamel's answer without ever considering the full Navier–Stokes equations at all.

In forward boundary layers the errors in the asymptotic expansions and the vorticity in the outer flow are multiples of $\exp(-\gamma y^2/\nu)$, where γ is a constant. For backward boundary layers, with $m = -1$, for which the fluid in the boundary layer has been flowing along a solid wall over an infinite distance, the error and the vorticity are of order $\exp(-\sqrt{2}y/\nu^{\frac{1}{2}})$. So they are still exponentially small, but not, after this long contact with a wall, as small as in a forward boundary layer. (For flow between non-parallel walls or equiangular spirals, this result is clear from boundary-layer solutions which have been known for a long time, but has never been commented on.)

6. Axisymmetric flow in converging passages. Converging flow in a cone

Backward axisymmetric boundary layers, and the flow in axisymmetric converging passages, may be discussed in the same way, but the results may easily be obtained from the two-dimensional results by using Mangler's transformation. Let r_0 be the distance of a point on a bounding streamline from the axis of symmetry. For the axisymmetric flow, the equation of continuity for a fluid of constant density and viscosity is written

$$\frac{\partial}{\partial x}(r_0 u)+\frac{\partial}{\partial y}(r_0 v) = 0 \tag{40}$$

in the boundary layer, and a stream function ψ is defined for which

$$r_0 u = \frac{\partial \psi}{\partial y}, \quad r_0 v = -\frac{\partial \psi}{\partial x}. \tag{41}$$

Use primes for a two-dimensional flow, for which

$$u' = \frac{\partial \psi'}{\partial y'}, \quad v' = -\frac{\partial \psi'}{\partial x'}. \tag{42}$$

r_0 is a given function of x. The boundary-layer equations in an axisymmetric flow are transformed into the equations of a two-dimensional boundary layer if we keep ρ, ν, p, and U the same, and put

$$x' = \frac{1}{l^2}\int_0^x r_0^2\,dx, \quad y' = \frac{r_0}{l}y, \quad \psi'(x',y') = \frac{1}{l}\psi(x,y). \tag{43}$$

Here l is an arbitrary fixed length, inserted merely to preserve dimensions. The resulting connexions between velocity components are

$$u' = u, \quad v' = \frac{l}{r_0}\left(v+\frac{y}{r_0}\frac{dr_0}{dx}u\right). \tag{44}$$

The boundary conditions are taken to be the same as before, although, as previously mentioned, the necessity that $(u/U-1)$ should be $o(Y'^{-N})$ as $Y'\to\infty$ has not been proved mathematically.

If now we again write s for distance along a streamline from the intersection of the bounding streamlines, and q for the resultant velocity, the condition in axisymmetric contracting passages which replaces (39) is seen to be

$$q \propto \left\{\int_0^s [r_0(s)]^2\,ds\right\}^{-1} \tag{45}$$

along the bounding streamlines. The shape of the streamlines that correspond dynamically to plane walls or symmetrically placed spiral walls in two dimensions has not yet been accurately found from this condition, and it is not yet certain that there is an accurate (as distinct from an approximate) solution.

If we consider converging flow in a right circular cone, the generators are straight, $r_0 = x\sin\alpha$, where α is the semi-vertical angle of the cone, and

$$x' = \frac{1}{3}\frac{x^3}{l^2}\sin^2\alpha, \quad y' = \frac{xy\sin\alpha}{l}. \tag{46}$$

For a simple sink flow a boundary layer will be impossible. For

$$U \propto \frac{1}{x^2} \text{ corresponds with } U \propto \frac{1}{x'^{\frac{2}{3}}}, \tag{47}$$

for which $m = -\frac{2}{3}$, $\beta = -4$. This is a case where the asymptotic error in f' behaves like ζ^{-8}, or Y^{-8}, and this algebraic error is taken to be inadmissible. In fact, for a circular cone

$$U \propto \frac{1}{x'} \text{ corresponds with } U \propto \frac{1}{x^3}, \tag{48}$$

and not $1/x^2$ as for a sink flow. (48) tells us how to start an expansion for small distances from the vertex of the cone. It turns out that there is no longer an irrotational flow outside a boundary layer, but (48) does tells us how the velocity just outside the boundary layer must vary with x very near the apex.

It has long been known (Harrison 1920) that purely radial flow cannot be a solution of the full Navier–Stokes equations for converging flow in a cone. However, a radial flow is a solution of Stokes's equations for creeping flow.

Consider the possibility of finding an axisymmetric solution of the full Navier–Stokes equations for converging flow in a cone, with a given volume flux $2\pi A$, and with purely radial flow at an infinitely great distance from the vertex, where the velocity will be infinitesimally small.

From dimensional reasoning, or from the equations of motion in spherical polar co-ordinates R, θ, where R is distance from the vertex and θ the angle with the axis of symmetry, together with the boundary conditions and equality of the volume flux to $2\pi A$, it is easy to see that, for a given semi-vertical angle α, the solution for Stokes's stream function ψ must be of the form

$$\frac{\psi}{A} = f(\xi, \theta), \quad \text{where} \quad \xi = \frac{\nu R}{A}. \tag{49}$$

Note that, in two dimensions, if A is the volume flux per unit breadth, it has one less dimension in length, and the corresponding result is

$$\frac{\psi}{A} = F\left(\frac{\nu}{A}, \theta\right), \tag{50}$$

with r absent, and this gives purely radial flow.

ξ is the inverse of a local Reynolds number which is very small when νR is very large—i.e. for large viscosities or at large distances, where the velocities are small. For large values of ξ there is an expansion of the function $f(\xi, \theta)$ in (49) which begins with the solution, $f_0(\mu)$, of Stokes's equations for creeping flow:

$$f = f_0(\mu) + \frac{1}{\xi} f_1(\mu) + \frac{1}{\xi^2} f_2(\mu) + \ldots, \quad \text{where} \quad \mu = \cos\theta. \tag{51}$$

The large Reynolds numbers occur for small values of ξ, near the apex of the cone. It is here that we expect an outer or core flow, with a boundary layer at the wall to bring the slip down to zero. In order that a boundary layer may exist, the first term in the expansion for small ξ must give a velocity proportional to R^{-3}. Hence the expansion of $f(\xi, \theta)$ for the outer or core flow near the apex must begin with a term in ξ^{-1}:

$$f = \frac{1}{\xi}[g_0(\mu) + \xi g_1(\mu) + \ldots]. \tag{52}$$

The corresponding expansion in the boundary layer (not just the first boundary-layer approximation) will be of the form

$$f = h_0(\tau) + \xi h_1(\tau) + \xi^2 h_2(\tau) + \ldots,$$

where

$$\tau = \frac{\cos\theta - \cos\alpha}{\xi}. \tag{53}$$

Note that here, very near the apex, the leading terms in U and therefore also in u in the boundary layer are $O(\nu^{-1})$. The elementary treatment of the boundary layer is therefore somewhat altered. For two-dimensional flow along a plane surface if $\partial/\partial x$ is taken of order unity, or for axisymmetric flow along a right conical surface if $\partial/\partial R$ is of order unity, the boundary-layer thickness is $O(\nu^{\frac{1}{2}}/|U|^{\frac{1}{2}})$. Here, then, the stretching factor in the boundary layer will be ν^{-1} instead of the usual $\nu^{-\frac{1}{2}}$. This explains the appearance of ν in the denominator in τ when the value of ξ is substituted from (49). It follows from an examination of the equations in spherical polar co-ordinates that the form for τ in (53) is a suitable form for a 'similarity' variable, and τ comes out to be a function of ξ and θ, as it should. Alternatively, the result that τ must be a function of ξ and θ may be used from the beginning (Ackerberg 1962). In addition to the order of the boundary-layer thickness, δ, note also that v is $O(U\delta)$, $\partial p/\partial x$ (or $\partial p/\partial R$) is $O(U^2)$, and $\partial p/\partial y$ (or $R^{-1}\,\partial p/\partial\theta$) is $O(U^2\delta)$, so here v is $O(1)$, $\partial p/\partial R$ is $O(\nu^{-2})$, and $R^{-1}\,\partial p/\partial\theta$ is $O(\nu^{-1})$.

The assumed expansions (52) and (53) are substituted in the full equation for ψ, in the first case with ξ and μ as variables, and in the second case with ξ and τ as variables, and like powers of ξ on the two sides of the equation are equated. The matching is done as $\tau \to \infty$ in (53) ($\xi \to 0$ and $\alpha - \theta$ not zero) and $\mu \to \cos\alpha$ in (52).

The volume flux $2\pi A$ is a constant, independent of ξ, and can come only from the second term in (52), the contribution to the flux from every other term being zero. In particular, the flux from the leading term must be zero, so we expect the radial flow to be outwards in part of a section very near the apex, and inwards in other parts of the same section; in other words, we expect, very near the apex, where ξ^{-1} is sufficiently large, a vortex region of closed streamlines.

The first approximation, $A\xi^{-1}g_0(\mu)$, to the stream function for the core flow is not independent of ν; as $\nu \to 0$, there is no limit. This is not the standard case of a region of axisymmetric vortex flow, and we cannot deduce that the vorticity should be proportional to the distance from the axis of symmetry; clearly it is not.

The suggested expansions have been studied and the calculations carried further by R. C. Ackerberg (1962) in his Harvard Ph.D. thesis (see also Ackerberg 1965). Unknown constants occur in (52) and (53) which, as often happens, can be found only by joining to the flow upstream. Logarithms have not yet been found necessary in (52) or (53), and it is not yet known if they will be necessary. Here the unknown constants and logarithms have not appeared simultaneously as they often do.

The suggested phenomenon will be difficult (if not impossible) to see experimentally; otherwise it would have been noted previously. Swirl must be absent. Surface tension effects must not be allowed, and certainly bubbles entering the opening must be avoided. The whole phenomenon takes place very near the apex, and it will be necessary to have a very small opening. There is a stagnation point on the axis of the cone at, say, $R = R_s$, and R_s must be large enough to allow the phenomenon to be seen; however, with this value of R, ξ^{-1} must be quite large, and this demands a small value of ν/A. So a small viscosity is required, and a large flux is needed even though the opening must be very small.. It is also important that in the vortical region near the apex R should be large enough for the actual ambient pressure near the hole (which will occur instead of the theoretical singularity at the vertex) not to disrupt the phenomenon. The production of the vortex near the apex must be due to the action of viscosity on each portion of the fluid over a sufficiently long time and therefore over a sufficiently long distance. In the mathematical theory the flow enters radially at a very large distance, and, although this requirement is probably otherwise not very stringent, yet, as Dr Ackerberg pointed out to me, if the fluid is supposed to enter the cone without vorticity a large distance from the apex will be needed if viscosity is to produce the vortical motion near the apex. In fact, whereas ξ must be small at the stagnation point on the axis and in the vortical region, it must be large at entry. So ν/A should be small, and R should be large at entry. Hence a long cone, or something equivalent, will be needed. Experiments have been made by Mr Binnie in Cambridge, England, but the phenomenon has not been observed in the experiments.

This research was supported by the U.S. Office of Naval Research under Contract NONR-1866(34).

REFERENCES

Ackerberg, R. C. 1962 Harvard Ph.D. Thesis (to be published).

Ackerberg, R. C. 1965 *J. Fluid Mech.* **21**, 47.

Coppel, W. A. 1960 *Phil. Trans.* A, **253**, 101–36.

Falkner, V. M. & Skan, S. W. 1930 *Aeronautical Research Committee, R. & M.* no. 1314.

Goldstein, S. 1939 *Proc. Camb. Phil. Soc.* **35**, 338–40.

Hamel, G. 1916 *Jahresbericht der Deutschen Mathematiker-Vereinigung*, **35**, 34–60.

Hardy, G. H. 1939 *Proc. Camb. Phil. Soc.* **35**, 652–3.

Harrison, W. J. 1920 *Proc. Camb. Phil. Soc.* **19**, 307–12.

Hartree, D. R. 1937 *Proc. Camb. Phil. Soc.* **33**, 223–39.

Iglisch, R. 1953 *Z.A.M.M.* **33**, 143–7.

Iglisch, R. 1954 *Z.A.M.M.* **34**, 441–3.

Kaplun, S. 1954 *J. Appl. Math. Phys.* **5**, 111–35.

Lagerstrom, P. A. & Cole, J. D. 1955 *J. Rat. Mech. Anal.* **4**, 817–82.

Mills, R. H. 1938 *J. Aero. Sci.* **5**, 325–7.

Serrin, J. 1962 Department of Mathematics, University of Minnesota, Lecture Notes on *Mathematical Aspects of Boundary Layer Theory.*

Stewartson, K. 1954 *Proc. Camb. Phil. Soc.* **50**, 454–65.

Weyl, H. 1942 *Ann. Math.* **43**, 381–407.

Whittaker, E. T. & Watson, G. N. 1920 *Modern Analysis* (3rd ed.). Cambridge University Press.